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# UNIVERSAL TORSORS OVER DEL PEZZO SURFACES AND RATIONAL POINTS

*by*

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ABSTRACT. — We discuss Manin’s conjecture concerning the distribution of rational points of bounded height on Del Pezzo surfaces, and its refinement by Peyre, and explain applications of universal torsors to counting problems. To illustrate the method, we provide a proof of Manin’s conjecture for the unique split singular quartic Del Pezzo surface with a singularity of type  $\mathbf{D}_4$ .

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## 1. Introduction

Let  $f \in \mathbb{Z}[x_0, \dots, x_n]$  be a non-singular form of degree  $d$ . By the circle method,

$$N(f, B) := \#\{\mathbf{x} \in \mathbb{Z}^{n+1}/\pm \mid \max_j(|x_j|) \leq B\} \sim c \cdot B^{n+1-d}$$

(where  $\mathbf{x} \in \mathbb{Z}^{n+1}/\pm$  means that we identify  $\mathbf{x}$  with  $-\mathbf{x} = (-x_0, \dots, -x_n)$ ) with  $c \in \mathbb{R}_{>0}$ , provided that  $n \geq 2^d \cdot (d-1)$ , and  $f(\mathbf{x}) = 0$  has solutions over all completions of  $\mathbb{Q}$  (see [Bir62]). Let  $X = X_f \subset \mathbb{P}^n$  be a smooth hypersurface over  $\mathbb{Q}$ , given by  $f(\mathbf{x}) = 0$ . It follows that

$$(1.1) \quad N(X, -K_X, B) = \#\{\mathbf{x} \in X(\mathbb{Q}) \mid H_{-K_X}(\mathbf{x}) \leq B\} \sim C \cdot B,$$

as  $B \rightarrow \infty$ . Here  $X(\mathbb{Q})$  is the set of rational points on  $X$ , represented by primitive vectors  $\mathbf{x} \in (\mathbb{Z}_{\text{prim}}^{n+1} \setminus 0)/\pm$  (i.e.,  $\mathbf{x} = (x_0, \dots, x_n)$  is identified with  $-\mathbf{x}$ , and there is no prime dividing all coordinates  $x_0, \dots, x_n$ ), and

$$(1.2) \quad H_{-K_X}(\mathbf{x}) := \max_j(|x_j|)^{n+1-d}, \quad \text{for } \mathbf{x} = (x_0, \dots, x_n) \in (\mathbb{Z}_{\text{prim}}^{n+1} \setminus 0)/\pm.$$

is the *anticanonical height* of a primitive representative.

In 1989 Manin initiated a program towards understanding connections between certain geometric invariants of algebraic varieties over number fields and their arithmetic properties, in particular, the distribution of rational points of bounded height, see [FMT89] and [BM90]. The main goal is an extension of the asymptotic formula (1.1) to other algebraic varieties of *small* degree, called Fano varieties, which are not necessarily isomorphic to hypersurfaces in projective space.

It became apparent, that in general, to obtain a geometric interpretation of asymptotic results, it may be necessary to restrict to appropriate Zariski open subsets of  $X$ . Otherwise, the number of rational points on a Zariski closed subset of lower dimension may dominate the total number of rational points; e.g., this phenomenon occurs for the surface (1.4) below where we will restrict to the complement of its lines. Furthermore, we may need to allow finite field extensions: while  $X(\mathbb{Q})$  might be empty,  $X(k)$  could still contain infinitely many points for some number field  $k$ .

Of particular interest are Del Pezzo surfaces (cf. [Man86]), e.g., cubic surfaces  $S_3 \subset \mathbb{P}^3$  or degree 4 surfaces  $S_4 := Q_1 \cap Q_2 \subset \mathbb{P}^4$ , where  $Q_1, Q_2$  are *quadrics* (defined by homogeneous equations of degree 2 in  $x_0, \dots, x_4$ ).

Geometrically, smooth Del Pezzo surfaces are obtained by blowing up  $\leq 8$  *general points*<sup>(1)</sup> in  $\mathbb{P}^2$ . Blowing up is a standard procedure in algebraic geometry (cf. [Har77, Section I.4]). The blow-up  $\pi : S' \rightarrow S$  of a surface  $S$  at a point  $p$  replaces  $p$  by a curve  $E$  in a particular way. We have  $S \setminus \{p\} \cong S' \setminus E$ , so  $S$  and  $S'$  are *birational*. In our situation, this shows that Del Pezzo surfaces are birational to  $\mathbb{P}^2$ , provided the ground field is algebraically closed.

We can think of *divisors* on blow-ups  $S$  of  $\mathbb{P}^2$  as formal sums of curves on  $S$ . Considering divisors up to a certain equivalence relation (see [Har77, Section II.6]) leads to the *Picard group*  $\text{Pic}(S)$  of divisor classes on  $S$ .

For two curves on  $S$  which intersect transversally, their *intersection number* is the number of intersection points. As explained in [Har77, Section V.1], this can be extended to arbitrary divisor classes, defining the non-degenerate *intersection form*  $(\cdot, \cdot)$  on  $\text{Pic}(S)$ . In particular, this defines the *self intersection number*  $(E, E)$  of (the class of) a curve  $E$ . Of special interest are irreducible curves for which this number is negative. We call them *exceptional curves*. For smooth Del Pezzo surfaces of degree 3 and 4, the exceptional curves are exactly the lines (in the standard embedding considered above), having self intersection number  $-1$ .

The singular Del Pezzo surfaces are obtained as follows: we blow up  $\mathbb{P}^2$  in special configurations of points (e.g., three points on a line). This results in a smooth surface  $\tilde{S}$  containing exceptional curves with self intersection number  $-2$  (called  $(-2)$ -curves; we do not permit to blow up points on  $(-2)$ -curves subsequently). Contracting the  $(-2)$ -curves gives a singular Del Pezzo surface  $S$  whose *minimal desingularization* is  $\tilde{S}$ . For the surface (1.4) below, more details can be found in Section 2.

For number fields, we say that a Del Pezzo surface is *split* if all of the exceptional curves are defined over that ground field; there exist *non-split* forms, some of which are not birational to  $\mathbb{P}^2$  over that ground field.

From now on, we work over  $\mathbb{Q}$ . Manin's conjecture in the special case of Del Pezzo surfaces can be formulated as follows.

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<sup>(1)</sup>no three points on a line, no six points on a curve of degree 2, no eight points with one of them singular on a curve of degree 3

CONJECTURE 1. — *Let  $S$  be a Del Pezzo surface with at most rational double points<sup>(2)</sup> over  $\mathbb{Q}$ . Then there exists a subset  $S^\circ \subset S$  which is dense and open in the Zariski topology such that*

$$(1.3) \quad N(S^\circ, -K_S, B) \sim c_{S,H} \cdot B(\log B)^{r-1},$$

as  $B \rightarrow \infty$ , where  $r$  is the rank of the Picard group of the minimal desingularization  $\tilde{S}$  of  $S$ , over  $\mathbb{Q}$ .

The constant  $c_{S,H}$  has been defined by Peyre [Pey95]; it should be non-zero if  $S(\mathbb{Q}) \neq \emptyset$ . Note that a line defined over  $\mathbb{Q}$  on a Del Pezzo surface such as  $S_3$  or  $S_4$  contributes  $\sim c \cdot B^2$  rational points to the counting function (for some positive constant  $c$ ). Thus it is expected that  $S^\circ$  is the complement to all lines defined over  $\mathbb{Q}$  (exceptional curves).

Table 1 gives an overview of current results towards Conjecture 1 for Del Pezzo surfaces. In Column 4 (“type of result”), “asymptotic” means that the analog of (1.3) is established, including the predicted value of the constant; “bounds” means that only upper and lower bounds of the expected order of magnitude  $B(\log B)^{r-1}$  with unknown constants are proved.

The paper [BT98] contains a proof of Manin’s conjecture for toric Fano varieties, including all smooth Del Pezzo surfaces of degree  $\geq 6$  and the  $3\mathbf{A}_2$  cubic surface<sup>(3)</sup>. This result also covers:

- all singular surfaces of degree  $\geq 7$  (i.e.,  $\mathbf{A}_1$  in degree 7 and 8),
- $\mathbf{A}_1$ ,  $2\mathbf{A}_1$ ,  $\mathbf{A}_2 + \mathbf{A}_1$  in degree 6,
- $2\mathbf{A}_1$ ,  $\mathbf{A}_2 + \mathbf{A}_1$  in degree 5,
- $4\mathbf{A}_1$ ,  $\mathbf{A}_2 + 2\mathbf{A}_1$ ,  $\mathbf{A}_3 + 2\mathbf{A}_1$  in degree 4.

Figure 1 shows all points of height  $\leq 50$  on the Cayley cubic surface (Example 14), which has four singularities of type  $\mathbf{A}_1$  and was considered in [HB03]. In Figure 2, we see points of height  $\leq 1000$  on the  $\mathbf{E}_6$  cubic surface ([Der05] and [BBD05]).

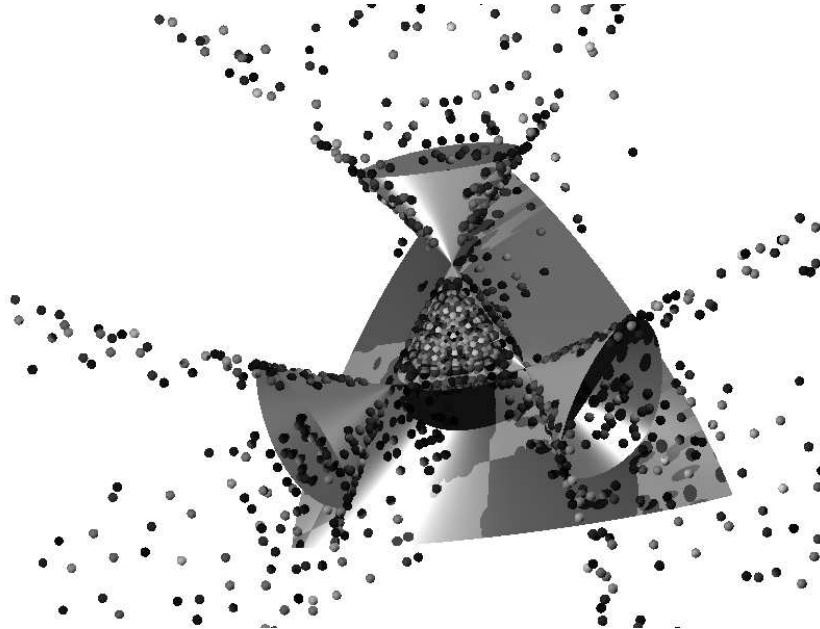
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<sup>(2)</sup>“mild” singularities which can be resolved by blow-ups to a curve whose irreducible components are isomorphic to  $\mathbb{P}^1$

<sup>(3)</sup>Singular Del Pezzo surfaces  $S$  will be labeled by the type (in the ADE-classification) and number of their singularities. The corresponding Dynkin diagram describes the number and intersection behaviour of the  $(-2)$ -curves on  $\tilde{S}$ .

degree	singularities	(non-)split	type of result	reference
$\geq 6$	–	split	asymptotic	[BT98]
5	–	split	asymptotic	[Bre02]
5	–	non-split	asymptotic	[BF04]
4	$D_5$	split	asymptotic	[CLT02], [BB] [BB05] this paper [Bro05]
4	$D_4$	non-split	asymptotic	
4	$D_4$	split	asymptotic	
4	$3A_1$	split	bounds	
3	$3A_2$	split	asymptotic	[BT98], [Bre98], ... [HB03] [Bro04] [Der05], [BBD05]
3	$4A_1$	split	bounds	
3	$D_4$	split	bounds	
3	$E_6$	split	asymptotic	

TABLE 1. Results for Del Pezzo surfaces

FIGURE 1. Points of height  $\leq 50$  on the Cayley cubic surface  $x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3 = 0$ .

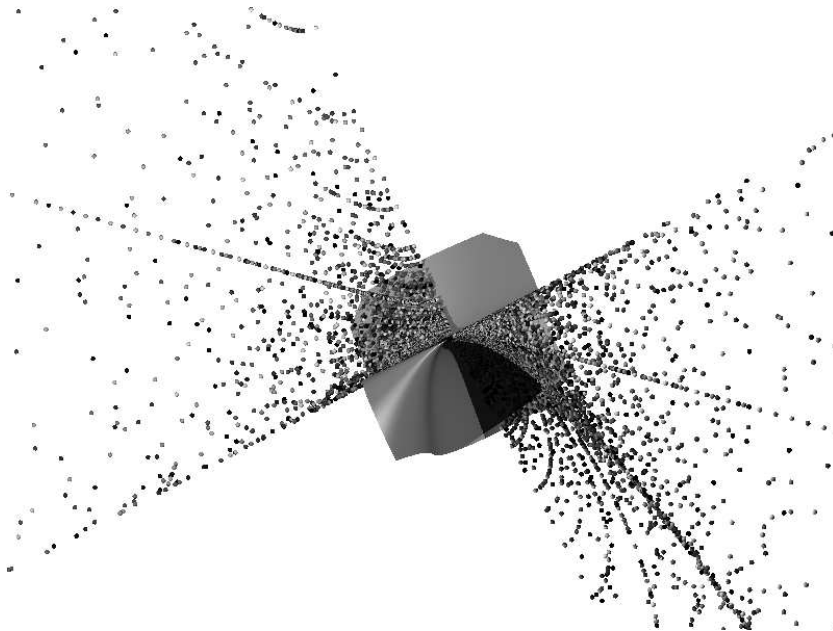


FIGURE 2. Points of height  $\leq 1000$  on the  $\mathbf{E}_6$  singular cubic surface  $x_1x_2^2 + x_2x_0^2 + x_3^3 = 0$  with  $x_0, x_2 > 0$ .

The proofs of Manin's conjecture proceed either via the height zeta function

$$Z(s) := \sum_{\mathbf{x} \in X^\circ(\mathbb{Q})} H_{-K_X}(\mathbf{x})^{-s},$$

whose analytic properties are related to the asymptotic (1.3) by Tauberian theorems, or via the lifting of the counting problem to the *universal torsor* – an auxiliary variety parametrizing rational points. The torsor approach has been developed by Colliot-Thélène and Sansuc in the context of the Brauer-Manin obstruction [CTS87] and applied to Manin's conjecture by Peyre [Pey98] and Salberger [Sal98].

In the simplest case of hypersurfaces  $X = X_f \subset \mathbb{P}^n$  over  $\mathbb{Q}$ , with  $n \geq 4$ , this is exactly the passage from rational vectors  $\mathbf{x} = (x_0, \dots, x_n)$ , modulo the diagonal action of  $\mathbb{Q}^*$ , to primitive lattice points  $(\mathbb{Z}_{\text{prim}}^{n+1} \setminus 0)/\pm$ . Geometrically, we have

$$\mathbb{A}^{n+1} \setminus 0 \xrightarrow{\mathbb{G}_m} \mathbb{P}^n \quad \text{and} \quad \mathcal{T}_X \xrightarrow{\mathbb{G}_m} X.$$

Here,  $\mathcal{T}_X$  is the hypersurface in  $\mathbb{A}^{n+1} \setminus 0$  defined by the form  $f$ , the 1-dimensional torus  $\mathbb{G}_m$  is interpreted as the Néron-Severi torus  $T_{\text{NS}}$ , i.e., an algebraic torus whose characters  $\mathfrak{X}^*(T_{\text{NS}})$  are isomorphic to the Picard group (lattice) of  $\mathbb{P}^n$ , resp.  $X$ , and the map is the natural quotient by its (diagonal) action. Rational points on the base are lifted to integral points on the torsor, modulo the action of the group of units  $T_{\text{NS}}(\mathbb{Z}) = \{\pm 1\}$ . The height inequality on the base  $H(\mathbf{x}) \leq B$  translates into the usual height inequality on the torsor (1.2).

In general, a torsor under an algebraic torus  $T$  is determined by a homomorphism  $\chi : \mathfrak{X}^*(T) \rightarrow \text{Pic}(X)$  to the Picard group of the underlying variety  $X$ ; the term *universal* is applied when  $\chi$  is an isomorphism.

However, for hypersurfaces in  $\mathbb{P}^3$ , or more generally for complete intersection surfaces (i.e.,  $S$  is the intersection of  $k$  hypersurfaces in  $\mathbb{P}^{k+2}$ ), the Picard group may have higher rank. For example, for split smooth cubic surfaces  $S = S_3 \subset \mathbb{P}^3$  the rank is 7, so that the dimension of the corresponding universal torsor  $\mathcal{T}_S$  is 9; for quartic Del Pezzo surfaces these are 6 and 8, respectively.

It is expected that the passage to universal torsors, which can be considered as natural *descent varieties*, will facilitate the proof of Manin's conjecture (Conjecture 1), at least for Del Pezzo surfaces. Rational points on  $S$  are lifted to certain integral points on  $\mathcal{T}_S$ , modulo the action of  $T_{\text{NS}}(\mathbb{Z}) = (\pm 1)^r$ , where  $r$  is the rank of  $\text{Pic}(S)$ , and the height inequality on  $S$  translates into appropriate inequalities on  $\mathcal{T}_S$ . This explains the interest in the projective geometry of torsors, and especially, in their equations. The explicit determination of these equations is an interesting algebro-geometric problem, involving tools from invariant theory and toric geometry.

In this note, we illustrate the torsor approach to asymptotics of rational points in the case of a particular singular surface  $S \subset \mathbb{P}^4$  of degree 4 given by:

$$(1.4) \quad x_0x_3 - x_1x_4 = x_0x_1 + x_1x_3 + x_2^2 = 0.$$

This is a split Del Pezzo surface, with a singularity of type  $\mathbf{D}_4$ .

**THEOREM 2.** — *The number of  $\mathbb{Q}$ -rational points of anticanonical height bounded by  $B$  on the complement  $S^\circ$  of the  $\mathbb{Q}$ -rational lines on  $S$  as in*

(1.4) *satisfies*

$$N(S^\circ, -K_S, B) = c_{S,H} \cdot B \cdot Q(\log B) + O(B(\log B)^3) \quad \text{as } B \rightarrow \infty,$$

where  $Q$  is a monic polynomial of degree 5, and

$$c_{S,H} = \frac{1}{34560} \cdot \omega_\infty \cdot \prod_p (1 - 1/p)^6 (1 + 6/p + 1/p^2)$$

is the constant predicted by Peyre [Pey95], with  $p$  running through all primes and

$$\omega_\infty = 3 \int \int \int_{\{(t,u,v) \in \mathbb{R}^3 \mid 0 \leq v \leq 1, |tv^2|, |v^2u|, |v(tv+u^2)|, |t(tv+u^2)| \leq 1\}} 1 \, dt \, du \, dv.$$

In [BB05], Manin's conjecture is proved for a non-split surface with a singularity of the same type. However, these results do not follow from each other.

In Section 2, we collect some facts about the geometric structure of  $S$ . In Section 3, we calculate the expected value of  $c_{S,H}$  and show that Theorem 2 agrees with Manin's conjecture.

In our case, the universal torsor is an affine hypersurface. In Section 4, we calculate its equation, stressing the relation with the geometry of  $S$ . We make explicit the coprimality and the height conditions. The method is more systematic than the derivation of torsor equations in [BB] and [BBD05], and should bootstrap to more complicated cases, e.g., other split Del Pezzo surfaces.

Note that our method gives coprimality conditions which are different from the ones in [BB] and [BBD05], but which are in a certain sense more natural: they are related to the set of points on  $\mathcal{T}_S$  which are *stable* with respect to the action of the Néron-Severi torus (in the sense of geometric invariant theory, c.f., [Dol03] and [HK00]). Our conditions involve only coprimality of certain pairs of variables, while the other method produces a mix of square-free variables and coprimality conditions.

In Section 5, we estimate the number of integral points on the universal torsor by iterating summations over the torsor variables and using results of elementary analytic number theory. Finally we arrive at Lemma 10, which is very similar in appearance to [BB, Lemma 10] and [Der05, Lemma 12]. In Section 6 we use familiar methods of height zeta functions to derive the exact asymptotic. We isolate the expected constant  $c_{S,H}$  and



finish the proof of Theorem 2. In Section 7 we write down examples of universal torsors for other Del Pezzo surfaces and discuss their geometry.

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## 2. Geometric background

In this section, we collect some geometric facts concerning the surface  $S$ . We show that Manin's conjecture for  $S$  is not a special case of available more general results for Del Pezzo surfaces.

LEMMA 3. — *The surface  $S$  has the following properties:*

- (1) *It has exactly one singularity of type  $\mathbf{D}_4$  at  $q = (0 : 0 : 0 : 0 : 1)$ .*
- (2)  *$S$  contains exactly two lines:*

$$E_5 = \{x_0 = x_1 = x_2 = 0\} \quad \text{and} \quad E_6 = \{x_1 = x_2 = x_3 = 0\},$$

*which intersect in  $q$ .*

- (3) *The projection from the line  $E_5$  is a birational map*

$$\begin{array}{ccc} \phi : S & \dashrightarrow & \mathbb{P}^2 \\ \mathbf{x} & \mapsto & (x_0 : x_2 : x_1) \end{array}$$

*which is defined outside  $E_5$ . It restricts to an isomorphism between*

$$S^\circ = S \setminus (E_5 \cup E_6) = \{\mathbf{x} \in S \mid x_1 \neq 0\} \quad \text{and} \quad \mathbb{A}^2 \cong \{(t : u : v) \mid v \neq 0\} \subset \mathbb{P}^2,$$

*whose inverse is the restriction of*

$$\begin{array}{ccc} \psi : \mathbb{P}^2 & \dashrightarrow & S, \\ (t : u : v) & \mapsto & (tv^2 : v^3 : v^2u : -v(tv + u^2) : -t(tv + u^2)) \end{array}$$

*Similar results hold for the projection from  $E_6$ .*

- (4) *The process of resolving the singularity  $q$  gives four exceptional curves  $E_1, \dots, E_4$  and produces the minimal desingularization  $\tilde{S}$ , which is also the blow-up of  $\mathbb{P}^2$  in five points.*

*Proof.* — Direct computations. □

It will be important to know the details of the sequence of five blow-ups of  $\mathbb{P}^2$  giving  $\tilde{S}$  as in Lemma 3(4):

In order to describe the points in  $\mathbb{P}^2$ , we need the lines

$$E_3 = \{v = 0\}, \quad A_1 = \{u = 0\}, \quad A_2 = \{t = 0\}$$

and the curve  $A_3 = \{tv + u^2 = 0\}$ .

LEMMA 4. — *The following five blow-ups of  $\mathbb{P}^2$  result in  $\tilde{S}$ :*

- *Blow up the intersection of  $E_3, A_1, A_3$ , giving  $E_2$ .*
- *Blow up the intersection of  $E_2, E_3, A_3$ , giving  $E_1$ .*
- *Blow up the intersection of  $E_1$  and  $A_3$ , giving  $E_4$ .*
- *Blow up the intersection of  $E_4$  and  $A_3$ , giving  $E_6$ .*
- *Blow up the intersection of  $E_3$  and  $A_2$ , giving  $E_5$ .*

Here, the order of the first four blow-ups is fixed, and the fifth blow-up can be done at any time.

The Dynkin diagram in Figure 3 describes the final configuration of divisors  $E_1, \dots, E_6, A_1, A_2, A_3$ . Here,  $A_1, A_2, A_3$  intersect at one point.

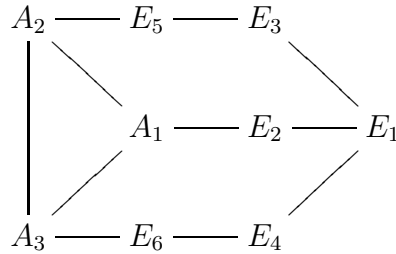


FIGURE 3. Extended Dynkin diagram

The quartic Del Pezzo surface with a singularity of type  $\mathbf{D}_4$  is not toric, and Manin's conjecture does not follow from the results of [BT98]. The  $\mathbf{D}_5$  example of [BB] is an equivariant compactification of  $\mathbb{G}_a^2$  (i.e.,  $S$  has a Zariski open subset isomorphic to  $\mathbb{A}^2$ , and the obvious action of  $\mathbb{G}_a^2$  on this open subset extends to  $S$ ), and thus a special case of [CLT02].

LEMMA 5. — *The quartic Del Pezzo surface with a singularity of type  $\mathbf{D}_4$  is a compactification of  $\mathbb{A}^2$ , but not an equivariant compactification of  $\mathbb{G}_a^2$ .*

*Proof.* — We follow the strategy of [HT04, Remark 3.3].

Consider the maps  $\phi, \psi$  as in Lemma 3(3). As  $\psi$  restricts to an isomorphism between  $\mathbb{A}^2$  and the open set  $S^\circ \subset S$ , the surface  $S$  is a compactification of  $\mathbb{A}^2$ .

If  $S$  were an equivariant compactification of  $\mathbb{G}_a^2$  then the projection  $\phi$  from  $E_5$  would be a  $\mathbb{G}_a^2$ -equivariant map, giving a  $\mathbb{G}_a^2$ -action on  $\mathbb{P}^2$ . The line  $\{v = 0\}$  would be invariant under this action. The only such action is the standard translation action

$$\begin{aligned} \tau : \quad \mathbb{P}^2 &\rightarrow \mathbb{P}^2, \\ (t : u : v) &\mapsto (t + \alpha v : u + \beta v : v). \end{aligned}$$

However, this action does not leave the linear series

$$(tv^2 : v^3 : v^2u : -v(tv + u^2) : -t(tv + u^2))$$

invariant, which can be seen after calculating

$$\begin{aligned} t(tv + u^2) &\mapsto (t + \alpha v)((t + \alpha v)v + (u + \beta v)^2) \\ &= t(tv + u^2) + 2\beta tuv + (\beta^2 + \alpha)tv^2 + \alpha v(tv + u^2) \\ &\quad + 2\alpha\beta v^2u + (\alpha\beta^2 + \alpha^2)v^3, \end{aligned}$$

since the term  $tuv$  does not appear in the original linear series.  $\square$

### 3. Manin's conjecture

LEMMA 6. — *Let  $S$  be the surface (1.4). Manin's conjecture for  $S$  states that the number of rational points of height  $\leq B$  outside the two lines is given by*

$$N(S^\circ, -K_S, B) \sim c_{S,H} \cdot B(\log B)^5,$$

where  $c_{S,H} = \alpha(S) \cdot \beta(S) \cdot \omega_H(S)$  with

$$\alpha(S) = (5! \cdot 4 \cdot 2 \cdot 3 \cdot 3 \cdot 2 \cdot 2)^{-1} = (34560)^{-1}$$

$$\beta(S) = 1$$

$$\omega_H(S) = \omega_\infty \cdot \prod_p (1 - 1/p)^6 (1 + 6/p + 1/p^2)$$

and

$$\omega_\infty = 3 \int \int \int_{\{(t,u,v) \in \mathbb{R}^3 \mid 0 \leq v \leq 1, |tv^2|, |v^2u|, |v(tv+u^2)|, |t(tv+u^2)| \leq 1\}} 1 \, dt \, du \, dv$$

*Proof.* — Since  $S$  is split over  $\mathbb{Q}$ , we have  $\mathrm{rk}(\mathrm{Pic}(\tilde{S})) = 6$ , and the expected exponent of  $\log B$  is 5. Further,  $\beta(S) = 1$ . The computation of  $c_{S,H}$  is done on the desingularization  $\tilde{S}$ . For the computation of  $\alpha(S)$ , observe that the effective cone of  $\tilde{S}$  in  $\mathrm{Pic}(\tilde{S})$  is simplicial (it is generated by the exceptional curves  $E_1, \dots, E_6$ , and their number equals the rank of  $\mathrm{Pic}(\tilde{S})$ ), and

$$-K_{\tilde{S}} = 4E_1 + 2E_2 + 3E_3 + 3E_4 + 2E_5 + 2E_6.$$

The calculation is analog to [Der05, Lemma 2] (see [Der06a] for its calculation in general). The constant  $\omega_H(S)$  is computed as in [BB, Lemma 1] and [Der05, Lemma 2].  $\square$

#### 4. The universal torsor

As explained above, the problem of counting rational points of bounded height on the surface  $S$  translates into a counting problem for certain integral points on the universal torsor, subject to coprimality and height inequalities. In the first part of this section, we describe these conditions in detail. They are obtained by a process of introducing new variables which are the greatest common divisors of other variables. Geometrically, this corresponds to the realization of  $\tilde{S}$  as a blow-up of  $\mathbb{P}^2$  in five points.

In the second part, we prove our claims.

The universal torsor  $\mathcal{T}_S$  of  $S$  is an open subset of the hypersurface in  $\mathbb{A}^9 = \mathrm{Spec} \mathbb{Z}[\eta_1, \dots, \eta_6, \alpha_1, \alpha_2, \alpha_3]$  defined by the equation

$$(4.1) \quad T(\boldsymbol{\eta}, \boldsymbol{\alpha}) = \alpha_1^2 \eta_2 + \alpha_2 \eta_3 \eta_5^2 + \alpha_3 \eta_4 \eta_6^2 = 0.$$

The projection  $\Psi : \mathcal{T}_S \rightarrow S$  is defined by

$$(4.2) \quad (\Psi^*(x_i)) = (\eta^{(2,1,2,1,2,0)} \alpha_2, \eta^{(4,2,3,3,2,2)} \eta^{(3,2,2,2,1,1)} \alpha_1, \eta^{(2,1,1,2,0,2)} \alpha_3, \alpha_2 \alpha_3),$$

where we use the notation  $\eta^{(n_1, n_2, n_3, n_4, n_5, n_6)} = \eta_1^{n_1} \eta_2^{n_2} \eta_3^{n_3} \eta_4^{n_4} \eta_5^{n_5} \eta_6^{n_6}$ .

The coprimality conditions can be derived from the extended Dynkin diagram (see Figure 3). Two variables are allowed to have a common factor if and only if the corresponding divisors ( $E_i$  for  $\eta_i$  and  $A_i$  for  $\alpha_i$ ) intersect (i.e., are connected by an edge in the diagram). Furthermore,  $\gcd(\alpha_1, \alpha_2, \alpha_3) > 1$  is allowed (corresponding to the fact that  $A_1, A_2, A_3$  intersect in one point).

We will show below that there is a bijection between rational points on  $S^\circ \subset S$  and integral points on an open subset of  $\mathcal{T}_S$ , subject to these coprimality conditions.

We will later refer to

$$(4.3) \quad \text{coprimality between } \eta_i \text{ as in Figure 3,}$$

$$(4.4) \quad \gcd(\alpha_1, \eta_1 \eta_3 \eta_4 \eta_5 \eta_6) = 1,$$

$$(4.5) \quad \gcd(\alpha_2, \eta_1 \eta_2 \eta_3 \eta_4 \eta_6) = 1,$$

$$(4.6) \quad \gcd(\alpha_3, \eta_1 \eta_2 \eta_3 \eta_4 \eta_5) = 1.$$

To count the number of  $\mathbf{x} \in S(\mathbb{Q})$  such that  $H(\mathbf{x}) \leq B$ , we must lift this condition to the universal torsor, i.e.,  $H(\Psi(\boldsymbol{\eta}, \boldsymbol{\alpha})) \leq B$ . This is the same as

$$|\eta^{(2,1,2,1,2,0)} \alpha_2| \leq B, \quad \dots, \quad |\alpha_2 \alpha_3| \leq B,$$

using the five monomials occurring in (4.2). These have no common factors, provided the coprimality conditions are fulfilled (direct verification).

It will be useful to write the height conditions as follows: let

$$X_0 = \left( \frac{\eta^{(4,2,3,3,2,2)}}{B} \right)^{1/3}, \quad X_1 = (B \eta^{(-1,-2,0,0,1,1)})^{1/3}, \quad X_2 = (B \eta^{(2,1,0,3,-2,4)})^{1/3}.$$

Then

$$(4.7) \quad |X_0^3| \leq 1$$

$$(4.8) \quad |X_0^2(\alpha_1/X_1)| \leq 1$$

$$(4.9) \quad \begin{aligned} |X_0^2(\alpha_2/X_2)| &\leq 1, & |X_0(X_0(\alpha_2/X_2) + (\alpha_1/X_1)^2)| &\leq 1, \\ |(\alpha_2/X_2)(X_0(\alpha_2/X_2) + (\alpha_1/X_1)^2)| &\leq 1 \end{aligned}$$

are equivalent to the five height conditions. Here we have used the torsor equation to eliminate  $\alpha_3$  because in our counting argument we will also use that  $\alpha_3$  is determined by the other variables.

We now prove the above claims.

LEMMA 7. — *The map  $\Psi$  gives a bijection between the set of points  $\mathbf{x}$  of  $S^\circ(\mathbb{Q})$  such that  $H(\mathbf{x}) \leq B$  and the set*

$$\mathcal{T}_1 := \left\{ (\boldsymbol{\eta}, \boldsymbol{\alpha}) \in \mathbb{Z}_{>0}^6 \times \mathbb{Z}^3 \left| \begin{array}{l} \text{equation (4.1),} \\ \text{coprimality (4.3), (4.4), (4.5), (4.6),} \\ \text{inequalities (4.7), (4.8), (4.6) hold} \end{array} \right. \right\}$$

*Proof.* — The map  $\psi$  of Lemma 3(3) induces a bijection

$$\psi_0 : (\eta_3, \alpha_1, \alpha_2) \mapsto (\eta_3^2 \alpha_2, \eta_3^3, \eta_3^2 \alpha_1, \eta_3 \alpha_3, \alpha_2 \alpha_3),$$

where  $\alpha_3 := -(\eta_3 \alpha_2 + \alpha_1^2)$ , i.e.,

$$T_0 := \alpha_1^2 + \eta_3 \alpha_2 + \alpha_3 = 0,$$

between

$$\{(\eta_3, \alpha_1, \alpha_2) \in \mathbb{Z}_{>0} \times \mathbb{Z}^2 \mid \gcd(\eta_3, \alpha_1, \alpha_2) = 1\} \quad \text{and} \quad S^\circ(\mathbb{Q}) \subset S(\mathbb{Q}).$$

The height function on  $S^\circ(\mathbb{Q})$  is given by

$$H(\psi_0(\eta_3, \alpha_1, \alpha_2)) = \frac{\max(|\eta_3^2 \alpha_2|, |\eta_3^3|, |\eta_3^2 \alpha_1|, |\eta_3 \alpha_3|, |\alpha_2 \alpha_3|)}{\gcd(\eta_3^2 \alpha_2, \eta_3^3, \eta_3^2 \alpha_1, \eta_3 \alpha_3, \alpha_2 \alpha_3)}.$$

The derivation of the torsor equation from the map  $\psi_0$  together with the coprimality conditions and the lifted height function is parallel to the blow-up process described in Lemma 4. More precisely, each line  $E_3, A_1, A_2$  in  $\mathbb{P}^2$  corresponds to a coordinate function  $\eta_3, \alpha_1, \alpha_2$  vanishing in one of the lines; the blow-up of the intersection of two divisors gives an exceptional curve  $E_i$ , corresponding to the introduction of a new variable  $\eta_i$  as the greatest common divisor of two old variables. Two divisors are disjoint if and only if the corresponding variables are coprime. This is summarized in Table 2.

Variables, Equations	Geometry
variables	divisors
initial variables	coordinate lines
$\eta_3, \alpha_1, \alpha_2$	$E_3, A_1, A_2$
taking gcd of two variables	blowing up intersection of divisors
new gcd-variable	exceptional curve
$\eta_2, \eta_1, \eta_4, \eta_6, \eta_5$	$E_2, E_1, E_4, E_6, E_5$
extra variable	extra curve
$\alpha_3$	$A_3$
starting relation	starting description
$\alpha_3 = -(\eta_3 \alpha_2 + \alpha_1^2)$	$A_3 = \{\eta_3 \alpha_2 + \alpha_1^2 = 0\}$
final relation	torsor equation
$\alpha_3 \eta_4 \eta_6^2 = -(\alpha_2 \eta_3 \eta_5^2 + \alpha_1^2 \eta_2)$	$\alpha_1^2 \eta_2 + \alpha_2 \eta_3 \eta_5^2 + \alpha_3 \eta_4 \eta_6^2 = 0$

TABLE 2. Dictionary between gcd-process and blow-ups

This plan will now be implemented in five steps; at each step, the map

$$\psi_i : \mathbb{Z}_{>0}^{i+1} \times \mathbb{Z}^3 \rightarrow S^\circ(\mathbb{Q})$$

gives a bijection between:

– the set of all  $(\eta_j, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_{>0}^{i+1} \times \mathbb{Z}^3$  satisfying certain coprimality conditions (described by the extended Dynkin diagram corresponding to the  $i$ -th blow-up of Lemma 4), an equation  $T_i$ ,

$$H(\psi_i(\eta_j, \alpha_j)) = \frac{\max_k (|\psi_i(\eta_j, \alpha_j)_k|)}{\gcd(\psi_i(\eta_j, \alpha_j)_k)} \leq B,$$

– the set of all  $\mathbf{x} \in S^\circ(\mathbb{Q})$  with  $H(\mathbf{x}) \leq B$ .

The steps are as follows:

(1) Let  $\eta_2 := \gcd(\eta_3, \alpha_1) \in \mathbb{Z}_{>0}$ . Then

$$\eta_3 = \eta_2 \eta'_3, \quad \alpha_1 = \eta_2 \alpha'_1, \quad \text{with } \gcd(\eta'_3, \alpha'_1) = 1.$$

Since  $\eta_2 \mid \alpha_3$ , we can write  $\alpha_3 = \eta_2 \alpha'_3$ . Then  $\alpha'_3 = -(\eta'_3 \alpha_2 + \eta_2 \alpha_1'^2)$ . After renaming the variables, we have

$$T_1 = \eta_2 \alpha_1'^2 + \eta_3 \alpha_2 + \alpha_3 = 0$$

and

$$\psi_1 : (\eta_2, \eta_3, \alpha_1, \alpha_2, \alpha_3) \mapsto (\eta_2 \eta_3^2 \alpha_2 : \eta_2^2 \eta_3^3 : \eta_2^2 \eta_3^2 \alpha_1 : \eta_2 \eta_3 \alpha_3 : \alpha_2 \alpha_3).$$

Here, we have eliminated the common factor  $\eta_2$  which occurred in all five components of the image. Below, we repeat the corresponding transformation at each step.

(2) Let  $\eta_1 := \gcd(\eta_2, \eta_3) \in \mathbb{Z}_{>0}$ . Then

$$\eta_2 = \eta_1 \eta'_2, \quad \eta_3 = \eta_1 \eta'_3, \quad \text{with } \gcd(\eta'_2, \eta'_3) = 1.$$

As  $\eta_1 \mid \alpha_3$ , we write  $\alpha_3 = \eta_1 \alpha'_3$ , and we obtain:

$$T_2 = \eta_2 \alpha_1'^2 + \eta_3 \alpha_2 + \alpha_3 = 0$$

and

$$\begin{aligned} \psi_2 : (\eta_1, \eta_2, \eta_3, \alpha_1, \alpha_2, \alpha_3) \mapsto \\ (\eta_1^2 \eta_2 \eta_3^2 \alpha_2 : \eta_1^4 \eta_2^2 \eta_3^3 : \eta_1^3 \eta_2^2 \eta_3^2 \alpha_1 : \eta_1^2 \eta_2 \eta_3 \alpha_3 : \alpha_2 \alpha_3). \end{aligned}$$

(3) Let  $\eta_4 := \gcd(\eta_1, \alpha_3) \in \mathbb{Z}_{>0}$ . Then

$$\eta_1 = \eta_4 \eta'_1, \quad \alpha_3 = \eta_4 \alpha'_3, \quad \text{with } \gcd(\eta'_1, \alpha'_3) = 1.$$

We get after removing ' again:

$$T_3 = \eta_2 \alpha_1^2 + \eta_3 \alpha_2 + \eta_4 \alpha_3 = 0$$

and

$$\begin{aligned} \psi_3 : (\eta_1, \eta_2, \eta_3, \eta_4, \alpha_1, \alpha_2, \alpha_3) \mapsto \\ (\eta_1^2 \eta_2 \eta_3^2 \eta_4 \alpha_2 : \eta_1^4 \eta_2^2 \eta_3^3 \eta_4^3 : \eta_1^3 \eta_2^2 \eta_3^2 \eta_4^2 \alpha_1 : \eta_1^2 \eta_2 \eta_3 \eta_4^2 \alpha_3 : \alpha_2 \alpha_3). \end{aligned}$$

(4) Let  $\eta_6 := \gcd(\eta_4, \alpha_3) \in \mathbb{Z}_{>0}$ . Then

$$\eta_4 = \eta_6 \eta'_4, \quad \alpha_3 = \eta_6 \alpha'_3, \quad \text{with } \gcd(\eta'_4, \alpha'_3) = 1.$$

We obtain

$$T_4 = \eta_2 \alpha_1^2 + \eta_3 \alpha_2 + \eta_4 \eta_6^2 \alpha_3 = 0$$

and

$$\begin{aligned} \psi_4 : (\eta_1, \eta_2, \eta_3, \eta_4, \eta_6, \alpha_1, \alpha_2, \alpha_3) \mapsto \\ (\eta_1^2 \eta_2 \eta_3^2 \eta_4 \alpha_2 : \eta_1^4 \eta_2^2 \eta_3^3 \eta_4^3 \eta_6^2 : \eta_1^3 \eta_2^2 \eta_3^2 \eta_4^2 \eta_6 \alpha_1 : \eta_1^2 \eta_2 \eta_3 \eta_4^2 \eta_6^2 \alpha_3 : \alpha_2 \alpha_3). \end{aligned}$$

(5) The final step is  $\eta_5 := \gcd(\eta_3, \alpha_2) \in \mathbb{Z}_{>0}$ , we could have done it earlier (just as the blow-up of the intersection of  $E_3, A_2$  in Lemma (4.2)). Then

$$\eta_3 = \eta_5 \eta'_3, \quad \alpha_2 = \eta_5 \alpha'_2, \quad \text{with } \gcd(\eta'_3, \alpha'_2) = 1.$$

We get

$$T_5 = \eta_2 \alpha_1^2 + \eta_3 \eta_5 \alpha_2 + \eta_4 \eta_6^2 \alpha_3 = 0$$

and

$$\begin{aligned} \psi_5 : (\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \alpha_1, \alpha_2, \alpha_3) \mapsto \\ (\eta_1^2 \eta_2 \eta_3^2 \eta_4 \eta_5^2 \alpha_2 : \eta_1^4 \eta_2^2 \eta_3^3 \eta_4^3 \eta_5^2 \eta_6^2 : \eta_1^3 \eta_2^2 \eta_3^2 \eta_4^2 \eta_5 \eta_6 \alpha_1 : \eta_1^2 \eta_2 \eta_3 \eta_4^2 \eta_6^2 \alpha_3 : \alpha_2 \alpha_3) \end{aligned}$$

We observe that at each stage the coprimality conditions correspond to intersection properties of the respective divisors. The final result is summarized in Figure 3, which encodes data from (4.3), (4.4), (4.5), (4.6).

Note that  $\psi_5$  is  $\Psi$  from (4.2). As mentioned above,  $\gcd(\psi_5(\eta_j, \alpha_j)_k)$  (over all five components of the image) is trivial by the coprimality conditions of Figure 3. Therefore,  $H(\psi_5(\boldsymbol{\eta}, \boldsymbol{\alpha})) \leq B$  is equivalent to (4.7), (4.8), (4.9).



Finally,  $T_5$  is the torsor equation  $T$  (4.1).  $\square$

## 5. Summations

In the first step, we estimate the number of  $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3$  which fulfill the torsor equation  $T$  (4.1) and the height and coprimality conditions. For fixed  $(\alpha_1, \alpha_2)$ , the torsor equation  $T$  has a solution  $\alpha_3$  if and only if the congruence

$$\alpha_1^2 \eta_2 + \alpha_2 \eta_3 \eta_5^2 \equiv 0 \pmod{\eta_4 \eta_6^2}$$

holds and the conditions on the height and coprimality are fulfilled.

We have already written the height conditions so that they do not depend on  $\alpha_3$ . For the coprimality, we must ensure that (4.5) and (4.6) are fulfilled.

As  $\gcd(\eta_3 \eta_5^2, \eta_4 \eta_6^2) = 1$ , we can find the multiplicative inverse  $c_1$  of  $\eta_3 \eta_5^2$  modulo  $\eta_4 \eta_6^2$ , so that

$$(5.1) \quad c_1 \eta_3 \eta_5^2 = 1 + c_2 \eta_4 \eta_6^2$$

for a suitable  $c_2$ . Choosing

$$(5.2) \quad \alpha_2 = c_3 \eta_4 \eta_6^2 - c_1 \alpha_1^2 \eta_2,$$

$$(5.3) \quad \alpha_3 = c_2 \alpha_1^2 \eta_2 - c_3 \eta_3 \eta_5^2$$

gives a solution of (4.1) for any  $c_3 \in \mathbb{Z}$ .

Without the coprimality conditions, the number of pairs  $(\alpha_2, \alpha_3)$  satisfying  $T$  and (4.9) would differ at most by  $O(1)$  from  $1/\eta_4 \eta_6^2$  of the length of the interval described by (4.9). However, the coprimality conditions (4.5) and (4.6) impose further restrictions on the choice of  $c_3$ . A slight complication arises from the fact that because of  $T$ , some of the conditions are fulfilled automatically once  $\eta, \alpha_1$  satisfy (4.3) and (4.4).

Conditions (4.3) imply that the possibilities for a prime  $p$  to divide more than one of the  $\eta_i$  are very limited. We distinguish twelve cases, listed in Column 2 of Table 3.

In Columns 4 and 5, we have denoted the relevant information for the divisibility of  $\alpha_2, \alpha_3$  by primes  $p$  which are divisors of the  $\eta_i$  in Column 2, but of no other  $\eta_j$ :

- “allowed” means that  $\alpha_i$  may be divisible by  $p$ .

case	$p \mid \dots$	$p \mid \alpha_1$	$p \mid \alpha_2$	$p \mid \alpha_3$
0	—	allowed	allowed	allowed
<i>i</i>	$\eta_1$	restriction	restriction	restriction
<i>ii</i>	$\eta_2$	allowed	restriction	automatically
<i>iii</i>	$\eta_3$	restriction	restriction	automatically
<i>iv</i>	$\eta_4$	restriction	automatically	restriction
<i>v</i>	$\eta_5$	restriction	allowed	automatically
<i>vi</i>	$\eta_6$	restriction	automatically	allowed
<i>vii</i>	$\eta_1, \eta_2$	restriction	restriction	automatically
<i>viii</i>	$\eta_1, \eta_3$	restriction	restriction	automatically
<i>ix</i>	$\eta_1, \eta_4$	restriction	automatically	restriction
<i>x</i>	$\eta_3, \eta_5$	restriction	restriction	automatically
<i>xi</i>	$\eta_4, \eta_6$	restriction	automatically	restriction

TABLE 3. Coprimality conditions

– “automatically” means that the conditions on the  $\eta_i$  and the other  $\alpha_j$  imply that  $p \nmid \alpha_i$ . These two cases do not impose conditions on  $c_3$  modulo  $p$ .

– “restriction” means that  $c_3$  is not allowed to be in a certain congruence class modulo  $p$  in order to fulfill the condition that  $p$  must not divide  $\alpha_i$ .

The information in the table is derived as follows:

– If  $p \mid \eta_3$ , then  $p \nmid c_2$  from (5.1), and  $p \nmid \alpha_1 \eta_2$  because of (4.3), (4.4), so by (5.3),  $p \nmid \alpha_3$  independently of the choice of  $c_3$ . Since  $p \nmid \eta_4 \eta_6^2$ , we see from (5.2) that  $p \mid \alpha_2$  for one in  $p$  subsequent choices of  $c_3$  which we must therefore exclude. This explains cases *iii* and *viii*.

– In case *vii*, the same is true for  $\alpha_2$ . More precisely, we see that we must exclude  $c_3 \equiv 0 \pmod{p}$ . By (5.3),  $p \nmid c_3$  implies that  $p \nmid \alpha_3$ , so we do not need another condition on  $c_3$ .

– In case *i*, we see that  $p \mid \alpha_2$  for one in  $p$  subsequent choices of  $c_3$ , and the same holds for  $\alpha_3$ . However, in this case,  $p$  cannot divide  $\alpha_2, \alpha_3$  for the same choice of  $c_3$ , as we can see by considering  $T$ : since  $p \nmid \alpha_1^2 \eta_2$ , it is impossible that  $p \mid \alpha_2, \alpha_3$ . Therefore, we must exclude two out of  $p$  subsequent choices of  $p$  in order to fulfill  $p \nmid \alpha_2, \alpha_3$ .

– In the other cases, the arguments are similar.

The number of  $(\alpha_2, \alpha_3) \in \mathbb{Z}^2$  subject to  $T$ , (4.5), (4.6), (4.9) equals the number of  $c_3$  such that  $\alpha_2, \alpha_3$  as in (5.2), (5.3) satisfy these conditions. This can be estimated as  $1/\eta_4\eta_6^2$  of the interval described by (4.9), multiplied by a product of local factors whose value can be read off from Columns 2, 4, 5 of Table 3: the divisibility properties of  $\eta_i$  by  $p$  determine whether zero, one or two out of  $p$  subsequent values of  $c_3$  have to be excluded. Different primes can be considered separately, and we define

$$\vartheta_{1,p} := \begin{cases} 1 - 2/p, & \text{case } i, \\ 1 - 1/p, & \text{cases } ii - iv, vi - xi, \\ 1, & \text{case } 0, v. \end{cases}$$

Let

$$\vartheta_1(\boldsymbol{\eta}) = \prod_p \vartheta_{1,p}$$

be the product of these local factors, and

$$(5.4) \quad g_1(u, v) = \int_{\{t \in \mathbb{R} \mid |tv^2|, |t(tv+u^2)|, |v(tv+u^2)| \leq 1\}} 1 \, dt.$$

Let  $\omega(n)$  denote the number of primes dividing  $n$ .

LEMMA 8. — *For fixed  $(\boldsymbol{\eta}, \alpha_1) \in \mathbb{Z}_{>0}^6 \times \mathbb{Z}$  as in (4.3), (4.4), (4.7), (4.8), the number of  $(\alpha_2, \alpha_3) \in \mathbb{Z}^2$  satisfying  $T$ , (4.5), (4.6), (4.9) is*

$$\mathcal{N}_1(\boldsymbol{\eta}, \alpha_1) = \frac{\vartheta_1(\boldsymbol{\eta})X_2}{\eta_4\eta_6^2} g_1(\alpha_1/X_1, X_0) + O(2^{\omega(\eta_1\eta_2\eta_3\eta_4\eta_6)}).$$

*The sum of error terms for all possible values of  $(\boldsymbol{\eta}, \alpha_1)$  is  $\ll B(\log B)^3$ .*

*Proof.* — The number of  $c_3$  such that the resulting  $\alpha_2, \alpha_3$  satisfy (4.9) differs from  $\frac{X_2}{\eta_4\eta_6^2} g_1(\alpha_1/X_1, X_0)$  by at most  $O(1)$ .

Each  $\vartheta_{1,p} \neq 1$  corresponds to a congruence condition on  $c_3$  imposed by one of the cases  $i - iv, vi - xi$ . For each congruence condition, the actual ratio of allowed  $c_3$  can differ at most by  $O(1)$  from the  $\vartheta_{1,p}$ . The total number of these primes  $p$  is

$$\omega(\eta_1\eta_2\eta_3\eta_4\eta_6) \ll 2^{\omega(\eta_1\eta_2\eta_3\eta_4\eta_6)},$$

which is independent of  $\eta_5$  since any prime dividing only  $\eta_5$  contributes a trivial factor (see case  $v$ ).

Using the estimate (4.8) for  $\alpha_1$  in the first step and ignoring (4.3) (4.4), which can only increase the error term, we obtain:

$$\sum_{\boldsymbol{\eta}} \sum_{\alpha_1} 2^{\omega(\eta_1 \eta_2 \eta_3 \eta_4 \eta_6)} \leq \sum_{\boldsymbol{\eta}} \frac{B \cdot 2^{\omega(\eta_1 \eta_2 \eta_3 \eta_4 \eta_6)}}{\eta^{(3,2,2,2,1,1)}} \ll B(\log B)^3.$$

Here, we use  $2^{\omega(n)} \ll_{\epsilon} n^{\epsilon}$  for the summations over  $\eta_1, \eta_2, \eta_3, \eta_4$ . For  $\eta_6$ , we employ

$$\sum_{n \leq x} 2^{\omega(n)} \ll x(\log x)$$

together with partial summation, contributing a factor  $(\log B)^2$ , while the summation over  $\eta_5$  gives another factor  $\log B$ .  $\square$

Next, we sum over all  $\alpha_1$  subject to the coprimality condition (4.4) and the height condition (4.8). Let

$$(5.5) \quad g_2(v) = \int_{\{u \in \mathbb{R} \mid |v^2 u| \leq 1\}} g_1(u, v) du$$

Similar to our discussion for  $\alpha_2, \alpha_3$ , the number of possible values for  $\alpha_1$  as in (4.8), while ignoring (4.4) for the moment, is  $X_1 g_2(X_0) + O(1)$ .

None of the coprimality conditions are fulfilled automatically, and only common factors with  $\eta_2$  are allowed (see Column 3 of Table 3). Therefore, each prime factor of  $\eta_1 \eta_3 \eta_4 \eta_5 \eta_6$  reduces the number of allowed  $\alpha_1$  by a factor of  $\vartheta_{2,p} = 1 - 1/p$  with an error of at most  $O(1)$ . For all other primes  $p$ , let  $\vartheta_{2,p} = 1$ , and let

$$\vartheta_2(\boldsymbol{\eta}) = \prod_p \vartheta_{2,p} \quad \text{and} \quad \vartheta(\boldsymbol{\eta}) = \begin{cases} \vartheta_1(\boldsymbol{\eta}) \cdot \vartheta_2(\boldsymbol{\eta}), & (4.3) \text{ holds} \\ 0, & \text{otherwise.} \end{cases}$$

LEMMA 9. — *For fixed  $\boldsymbol{\eta} \in \mathbb{Z}_{>0}^6$  as in (4.3), (4.7), the sum of  $\mathcal{N}_1(\boldsymbol{\eta}, \alpha_1)$  over all  $\alpha_1 \in \mathbb{Z}$  satisfying (4.4), (4.8) is*

$$\mathcal{N}_2(\boldsymbol{\eta}) := \frac{\vartheta(\boldsymbol{\eta}) X_1 X_2}{\eta_4 \eta_6^2} g_2(X_0) + \mathcal{R}_2(\boldsymbol{\eta}),$$

where the sum of error terms  $\mathcal{R}_2(\boldsymbol{\eta})$  over all possible  $\boldsymbol{\eta}$  is  $\ll B \log B$ .

*Proof.* — Let

$$\mathcal{N}(b_1, b_2) = \vartheta_1(\boldsymbol{\eta}) \cdot \#\{\alpha_1 \in [b_1, b_2] \mid \gcd(\alpha_1, \eta_1 \eta_3 \eta_4 \eta_5 \eta_6) = 1\}.$$

Using Möbius inversion, this is estimated as

$$\mathcal{N}(b_1, b_2) = \vartheta_1(\boldsymbol{\eta}) \cdot \vartheta_2(\boldsymbol{\eta}) \cdot (b_2 - b_1) + \mathcal{R}(b_1, b_2)$$

with  $\mathcal{R}(b_1, b_2) = O(2^{\omega(\eta_1 \eta_3 \eta_4 \eta_5 \eta_6)})$ . By partial summation,

$$\mathcal{N}_2(\boldsymbol{\eta}) = \frac{\vartheta(\boldsymbol{\eta}) X_1 X_2}{\eta_4 \eta_6^2} g_2(X_0) + \mathcal{R}_2(\boldsymbol{\eta})$$

with

$$\mathcal{R}_2(\boldsymbol{\eta}) = \frac{-X_2}{\eta_4 \eta_6^2} \int_{\{u \mid X_0^2 u \leq 1\}} (D_1 g_1)(u, X_0) \mathcal{R}(-X_1/X_0^2, X_1 u) du$$

where  $D_1 g_1$  is the partial derivative of  $g_1$  with respect to the first variable. Using the above bound for  $\mathcal{R}(b_1, b_2)$ , we obtain:

$$\mathcal{R}_2(\boldsymbol{\eta}) \ll \frac{X_2}{\eta_4 \eta_6^2} 2^{\omega(\eta_1 \eta_3 \eta_4 \eta_5 \eta_6)}.$$

Summing this over all  $\boldsymbol{\eta}$  as in (4.7) while ignoring (4.3) which can only enlarge the sum, we obtain:

$$\sum_{\boldsymbol{\eta}} \mathcal{R}_2(\boldsymbol{\eta}) \ll \sum_{\boldsymbol{\eta}} \frac{X_2 \cdot 2^{\omega(\eta_1 \eta_3 \eta_4 \eta_5 \eta_6)}}{\eta_4 \eta_6^2 X_0^2} = \sum_{\boldsymbol{\eta}} \frac{B \cdot 2^{\omega(\eta_1 \eta_3 \eta_4 \eta_5 \eta_6)}}{\eta^{(2,1,2,2,2,2)}} \ll B \log B$$

In the first step, we use  $X_0 \leq 1$ .  $\square$

Let

$$\Delta(n) = B^{-2/3} \sum_{\eta_i, \eta^{(4,2,3,3,2,2)}=n} \frac{\vartheta(\boldsymbol{\eta}) X_1 X_2}{\eta_4 \eta_6^2} = \sum_{\eta_i, \eta^{(4,2,3,3,2,2)}=n} \frac{\vartheta(\boldsymbol{\eta}) (\eta^{(4,2,3,3,2,2)})^{1/3}}{\eta^{(1,1,1,1,1,1)}}.$$

In view of Lemma 7, the number of rational points of bounded height on  $S^\circ$  can be estimated by summing the result of Lemma 9 over all suitable  $\boldsymbol{\eta}$ . The error term is the combination of the error terms in Lemmas 8 and 9.

LEMMA 10. — *We have*

$$N(S^\circ, -K_S, B) = B^{2/3} \sum_{n \leq B} \Delta(n) g_2((n/B)^{1/3}) + O(B(\log B)^3).$$

## 6. Completion of the proof

We need an estimate for

$$M(t) := \sum_{n \leq t} \Delta(t).$$

Consider the Dirichlet series  $F(s) := \sum_{n=1}^{\infty} \Delta(n)n^{-s}$ . Using

$$F(s + 1/3) = \sum_{\boldsymbol{\eta}} \frac{\vartheta(\boldsymbol{\eta})}{\eta_1^{4s+1} \eta_2^{2s+1} \eta_3^{3s+1} \eta_4^{3s+1} \eta_5^{2s+1} \eta_6^{2s+1}},$$

we write  $F(s + 1/3) = \prod_p F_p(s + 1/3)$  as its Euler product. To obtain  $F_p(s + 1/3)$  for a prime  $p$ , we need to restrict this sum to the terms in which all  $\eta_i$  are powers of  $p$ . Note that  $\vartheta(\boldsymbol{\eta})$  is non-zero if and only if the divisibility of  $\eta_i$  by  $p$  falls into one of the twelve cases described in Table 3. The value of  $\vartheta(\boldsymbol{\eta})$  only depends on these cases.

Writing  $F_p(s + 1/3) = \sum_{i=1}^{11} F_{p,i}(s + 1/3)$ , we have for example:

$$\begin{aligned} F_{p,0}(s + 1/3) &= 1, \\ F_{p,1}(s + 1/3) &= \sum_{j=1}^{\infty} \frac{(1 - 1/p)(1 - 2/p)}{p^{j(4s+1)}} = \frac{(1 - 1/p)(1 - 2/p)}{p^{4s+1} - 1}, \\ F_{p,7}(s + 1/3) &= \sum_{j,k=1}^{\infty} \frac{(1 - 1/p)^2}{p^{j(4s+1)} p^{k(2s+1)}} = \frac{(1 - 1/p)^2}{(p^{4s+1} - 1)(p^{2s+1} - 1)}. \end{aligned}$$

The other cases are similiar, giving

$$\begin{aligned} F_p(s + 1/3) &= 1 + \frac{1 - 1/p}{p^{4s+1} - 1} \left( (1 - 2/p) + \frac{1 - 1/p}{p^{2s+1} - 1} + 2 \frac{1 - 1/p}{p^{3s+1} - 1} \right) \\ &\quad + \frac{1 - 1/p}{p^{2s+1} - 1} + 2 \frac{(1 - 1/p)^2}{p^{3s+1} - 1} + 2 \frac{1 - 1/p}{p^{2s+1} - 1} + 2 \frac{(1 - 1/p)^2}{(p^{2s+1} - 1)^2}. \end{aligned}$$

Defining

$$E(s) := \zeta(4s+1)\zeta(3s+1)^2\zeta(2s+1)^3 \quad \text{and} \quad G(s) := F(s+1/3)/E(s),$$

we see as in [Der05] that the residue of  $F(s)t^s/s$  at  $s = 1/3$  is

$$\text{Res}(t) = \frac{3G(0)t^{1/3}Q_1(\log t)}{5! \cdot 4 \cdot 2 \cdot 3 \cdot 3 \cdot 2 \cdot 2}$$

for a monic  $Q_1 \in \mathbb{R}[x]$  of degree 5. By Lemma 6,  $\alpha(S) = \frac{1}{5! \cdot 4 \cdot 2 \cdot 3 \cdot 3 \cdot 2 \cdot 2}$ . By a Tauberian argument as in [Der05, Lemma 13]:

LEMMA 11. —  $M(t) = \text{Res}(t) + O(t^{1/3-\delta})$  for some  $\delta > 0$ .

By partial summation,

$$\sum_{n \leq B} \Delta(n) g_2((n/B)^{1/3}) = \alpha(S) G(0) B^{1/3} Q(\log B) \cdot 3 \int_0^1 g_2(v) dv + O(B^{\frac{1}{3}-\delta})$$

for a monic polynomial  $Q$  of degree 5. We identify  $\omega_H(S)$  from

$$G(0) = \prod_p \left(1 - \frac{1}{p}\right)^6 \left(1 + \frac{6}{p} + \frac{1}{p^2}\right), \quad \text{and } \omega_\infty = 3 \int_0^1 g_2(v) dv.$$

Together with Lemma 10, this completes the proof of Theorem 2.

## 7. Equations of universal torsors

The simplest universal torsors are those which can be realized as Zariski open subsets of the affine space. This happens if and only if the Del Pezzo surface is toric.

EXAMPLE 12. — There are 20 types of singular Del Pezzo surfaces of degree  $d \geq 3$  whose universal torsor is an open subset of a hypersurface in  $\mathbb{A}^{13-d}$ . For one example of each type<sup>(4)</sup>, the equation defining the universal torsor is listed in the following table. More details can be

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<sup>(4)</sup>For the cubic  $\mathbf{D}_4$  case, the universal torsor of a different example is calculated in [HT04, Section 4].

found in [Der06d].

degree	singularities	# of lines	defining equation
6	$\mathbf{A}_1$	3	$\eta_2\alpha_1 + \eta_3\alpha_2 + \eta_4\alpha_3$
6	$\mathbf{A}_2$	2	$\eta_2\alpha_1^2 + \eta_3\alpha_2 + \eta_4\alpha_3$
5	$\mathbf{A}_1$	7	$\eta_2\eta_6 + \eta_3\eta_7 + \eta_4\eta_8$
5	$\mathbf{A}_2$	4	$\eta_2\eta_5^2\eta_6 + \eta_3\alpha_1 + \eta_4\alpha_2$
5	$\mathbf{A}_3$	2	$\eta_1\alpha_1^2 + \eta_3\eta_4^2\alpha_2 + \eta_5\alpha_3$
5	$\mathbf{A}_4$	1	$\eta_1^2\eta_2\alpha_1^3 + \eta_4\alpha_2^2 + \eta_5\alpha_3$
4	$3\mathbf{A}_1$	6	$\eta_4\eta_5 + \eta_1\eta_6\eta_7 + \eta_8\eta_9$
4	$\mathbf{A}_2 + \mathbf{A}_1$	6	$\eta_5\eta_7 + \eta_1\eta_3\eta_9^2 + \eta_6\eta_8$
4	$\mathbf{A}_3$	5	$\eta_5\alpha + \eta_1\eta_4^2\eta_7 + \eta_3\eta_6^2\eta_8$
4	$\mathbf{A}_3 + \mathbf{A}_1$	3	$\eta_6\alpha_2 + \eta_7\alpha_1 + \eta_1\eta_3\eta_4^2\eta_5^3$
4	$\mathbf{A}_4$	3	$\eta_5\alpha_1 + \eta_1\alpha_2^2 + \eta_3\eta_4^2\eta_6^3\eta_7$
4	$\mathbf{D}_4$	2	$\eta_3\eta_5^2\alpha_2 + \eta_4\eta_6^2\alpha_3 + \eta_2\alpha_1^2$
4	$\mathbf{D}_5$	1	$\eta_3\alpha_1^2 + \eta_2\eta_6^2\alpha_3 + \eta_4\eta_5^2\alpha_2^3$
3	$\mathbf{D}_4$	6	$\eta_2\eta_5^2\eta_8 + \eta_3\eta_6^2\eta_9 + \eta_4\eta_7^2\eta_{10}$
3	$\mathbf{A}_3 + 2\mathbf{A}_1$	5	$\eta_4\eta_6^2\eta_{10} + \eta_1\eta_2\eta_7^2 + \eta_8\eta_9$
3	$2\mathbf{A}_2 + \mathbf{A}_1$	5	$\eta_3\eta_5\eta_7^2 + \eta_1\eta_6\eta_8 + \eta_9\eta_{10}$
3	$\mathbf{A}_4 + \mathbf{A}_1$	4	$\eta_1\eta_5\eta_8^2 + \eta_3\eta_4^2\eta_6^3\eta_9 + \eta_7\alpha$
3	$\mathbf{D}_5$	3	$\eta_2\eta_6^2\alpha_2 + \eta_4\eta_5^2\eta_7^3\eta_8 + \eta_3\alpha_1^2$
3	$\mathbf{A}_5 + \mathbf{A}_1$	2	$\eta_1^3\eta_2^2\eta_3\eta_7^4\eta_8 + \eta_5\alpha_1^2 + \eta_6\alpha_2$
3	$\mathbf{E}_6$	1	$\eta_4^2\eta_5\eta_7^3\alpha_3 + \eta_2\alpha_2^2 + \eta_1^2\eta_3\alpha_1^3$

EXAMPLE 13 (Cubic surface with  $\mathbf{A}_1 + \mathbf{A}_3$  singularities)

This surface has 7 lines, 4 additional variables correspond to exceptional curves of the desingularization. Its 9-dimensional universal torsor is a Zariski open subset of a complete intersection in

$$\mathbb{A}^{11} = \text{Spec } \mathbb{Z}[\eta_0, \dots, \eta_3, \mu_0, \dots, \mu_6]$$

given by

$$\eta_1\eta_2\mu_1\mu_2 + \mu_4\mu_6 + \mu_3\mu_5 = 0 \quad \text{and} \quad \eta_0\eta_1\mu_2^2 + \eta_3\mu_5\mu_6 + \mu_0\mu_1 = 0.$$

See [Der06b] for more details.

There are examples of universal torsors which are not complete intersections, but have still been successfully used in the context of Manin's conjecture:



EXAMPLE 14 (Cayley cubic). — The Cayley cubic surface

$$x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3 = 0$$

(Figure 1) is a split singular cubic surface with four singularities  $q_1, \dots, q_4$  of type  $\mathbf{A}_1$  and nine lines. It is the blow-up of  $\mathbb{P}^2$  in the 6 intersection points of 4 lines in general position. The universal torsor is an open subvariety of the variety in

$$\mathbb{A}^{13} = \operatorname{Spec} \mathbb{Z}[v_{12}, v_{13}, v_{14}, y_1, y_2, y_3, y_4, z_{12}, z_{13}, z_{14}, z_{23}, z_{24}, z_{25}]$$

defined by six equations of the form

$$z_{ik}z_{il}y_j + z_{jk}z_{jl}y_i = z_{ij}v_{ij}$$

and three equations of the form

$$v_{ij}v_{ik} = z_{il}^2y_jy_k - z_{jk}^2y_iy_l,$$

where  $\{i, j, k, l\} = \{1, 2, 3, 4\}$  and

$$z_{ij} = z_{ji}, \quad v_{ij} = v_{ji}, \quad \text{and} \quad v_{ij} = -v_{kl}.$$

See [Der06b] for a proof. The variables  $y_i$  correspond to the four exceptional curves  $E_i$  obtained by blowing up  $q_i$ ,  $z_{ij}$  correspond to the six lines  $m_{ij}$  through two of the singularities, and  $v_{ij}$  correspond to the other three lines  $\ell_{ij}$ . The first six equations can be interpreted in connection with the projection from  $m_{ij}$ , and the other three equations are connected to the projection from  $\ell_{ij}$ .

Upper and lower bounds of the expected order of magnitude have been established in [HB03].

EXAMPLE 15 (Smooth degree 5 Del Pezzo surface)

The blow-up of  $\mathbb{P}^2$  in

$$(1 : 0 : 0), \quad (0 : 1 : 0), \quad (0 : 0 : 1), \quad (1 : 1 : 1)$$

is a split smooth Del Pezzo surface of degree 5. Its universal torsor is an open subset of the variety defined by the following five equations in ten

variables:

$$\begin{aligned}
\lambda_{12}\eta_2 - \lambda_{13}\eta_3 + \lambda_{14}\eta_4 &= 0 \\
\lambda_{12}\eta_1 - \lambda_{23}\eta_3 + \lambda_{24}\eta_4 &= 0 \\
\lambda_{13}\eta_1 - \lambda_{23}\eta_2 + \lambda_{34}\eta_4 &= 0 \\
\lambda_{14}\eta_1 - \lambda_{24}\eta_2 + \lambda_{34}\eta_3 &= 0 \\
\lambda_{12}\lambda_{34} - \lambda_{13}\lambda_{24} + \lambda_{14}\lambda_{23} &= 0
\end{aligned}$$

The asymptotic formula (1.3) has been established in [Bre02].

To illustrate some of the difficulties in proving Conjecture 1 for a smooth split cubic surface, we now write down equations for its universal torsor (up to radical).

EXAMPLE 16 (Smooth cubic surfaces). — Let  $S$  be the blow-up of  $\mathbb{P}^2$  in

$$(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : 1 : 1), (1 : a : b), (1 : c : d),$$

in general position. Conjecturally, the universal torsor is an open subset of the intersection of 81 quadrics in 27-dimensional space  $\text{Spec } \mathbb{Z}[\eta_i, \mu_{i,j}, \lambda_i]$ , where

- $\eta_1, \dots, \eta_6$  correspond to the preimages of the points,
  - $\mu_{i,j}$  ( $i < j \in \{1, \dots, 6\}$ ) correspond to the 15 lines  $m_{i,j}$  through two of the points,
  - $\lambda_1, \dots, \lambda_6$  correspond to the conics  $Q_i$  through five of the six points,
- and relations arise from conic bundle structures on  $S$ . Batyrev and Popov proved that the above variables are indeed generators and that the relations give the universal torsor, up to radical [BP04].

We now write down these equations explicitly (see [Der06c] for more details). The 81 defining quadrics occur in sets of three. These 27 triples correspond to projections from the 27 lines on  $S$ . We use

$$E := (b-1)(c-1) - (a-1)(d-1) \quad \text{and} \quad F := bc - ad$$

to simplify the equations.

$$\begin{aligned}
q_{Q_1,1} &= -\eta_2\mu_{1,2} - \eta_3\mu_{1,3} + \eta_4\mu_{1,4} \\
q_{Q_1,2} &= -a\eta_2\mu_{1,2} - b\eta_3\mu_{1,3} + \eta_5\mu_{1,5} \\
q_{Q_1,2} &= -c\eta_2\mu_{1,2} - d\eta_3\mu_{1,3} + \eta_6\mu_{1,6}
\end{aligned}$$

$$\begin{aligned}
q_{Q_2,1} &= \eta_1\mu_{1,2} - \eta_3\mu_{2,3} + \eta_4\mu_{2,4} \\
q_{Q_2,2} &= \eta_1\mu_{1,2} - b\eta_3\mu_{2,3} + \eta_5\mu_{2,5} \\
q_{Q_2,3} &= \eta_1\mu_{1,2} - d\eta_3\mu_{2,3} + \eta_6\mu_{2,6}
\end{aligned}$$

$$\begin{aligned}
q_{Q_3,1} &= \eta_1\mu_{1,3} + \eta_2\mu_{2,3} + \eta_4\mu_{3,4} \\
q_{Q_3,2} &= \eta_1\mu_{1,3} + a\eta_2\mu_{2,3} + \eta_5\mu_{3,5} \\
q_{Q_3,3} &= \eta_1\mu_{1,3} + c\eta_2\mu_{2,3} + \eta_6\mu_{3,6}
\end{aligned}$$

$$\begin{aligned}
q_{Q_4,1} &= \eta_1\mu_{1,4} + \eta_2\mu_{2,4} + \eta_3\mu_{3,4} \\
q_{Q_4,2} &= (1-b)\eta_1\mu_{1,4} + (a-b)\eta_2\mu_{2,4} + \eta_5\mu_{4,5} \\
q_{Q_4,3} &= (1-d)\eta_1\mu_{1,4} + (c-d)\eta_2\mu_{2,4} + \eta_6\mu_{4,6}
\end{aligned}$$

$$\begin{aligned}
q_{Q_5,1} &= 1/b\eta_1\mu_{1,5} + a/b\eta_2\mu_{2,5} + \eta_3\mu_{3,5} \\
q_{Q_5,2} &= (1-b)/b\eta_1\mu_{1,5} + (a-b)/b\eta_2\mu_{2,5} + \eta_4\mu_{4,5} \\
q_{Q_5,3} &= (b-d)/b\eta_1\mu_{1,5} + F/b\eta_2\mu_{2,5} + \eta_6\mu_{5,6}
\end{aligned}$$

$$\begin{aligned}
q_{Q_6,1} &= 1/d\eta_1\mu_{1,6} + c/d\eta_2\mu_{2,6} + \eta_3\mu_{3,6} \\
q_{Q_6,2} &= (1-d)/d\eta_1\mu_{1,6} + (c-d)/d\eta_2\mu_{2,6} + \eta_4\mu_{4,6} \\
q_{Q_6,3} &= (b-d)/d\eta_1\mu_{1,6} + F/d\eta_2\mu_{2,6} + \eta_5\mu_{5,6}
\end{aligned}$$

$$\begin{aligned}
q_{m_{1,2},1} &= \mu_{4,5}\mu_{3,6} - \mu_{3,5}\mu_{4,6} + \mu_{3,4}\mu_{5,6} \\
q_{m_{1,2},2} &= (b-d)\mu_{3,5}\mu_{4,6} + (d-1)\mu_{3,4}\mu_{5,6} + \eta_2\lambda_1 \\
q_{m_{1,2},3} &= F\mu_{3,5}\mu_{4,6} + a(d-c)\mu_{3,4}\mu_{5,6} + \eta_1\lambda_2
\end{aligned}$$

$$\begin{aligned}
q_{m_{1,3},1} &= \mu_{4,5}\mu_{2,6} - \mu_{2,5}\mu_{4,6} + \mu_{2,4}\mu_{5,6} \\
q_{m_{1,3},2} &= (c-a)\mu_{2,5}\mu_{4,6} + (1-c)\mu_{2,4}\mu_{5,6} + \eta_3\lambda_1 \\
q_{m_{1,3},3} &= -F\mu_{2,5}\mu_{4,6} + b(c-d)\mu_{2,4}\mu_{5,6} + \eta_1\lambda_3
\end{aligned}$$

$$\begin{aligned}
q_{m_{2,3},1} &= \mu_{4,5}\mu_{1,6} - \mu_{1,5}\mu_{4,6} + \mu_{1,4}\mu_{5,6} \\
q_{m_{2,3},2} &= (a-c)\mu_{1,5}\mu_{4,6} + a(c-1)\mu_{1,4}\mu_{5,6} + \eta_3\lambda_2 \\
q_{m_{2,3},3} &= (b-d)\mu_{1,5}\mu_{4,6} + b(d-1)\mu_{1,4}\mu_{5,6} + \eta_2\lambda_3
\end{aligned}$$

$$\begin{aligned}
q_{m_{1,4},1} &= \mu_{3,5}\mu_{2,6} - \mu_{2,5}\mu_{3,6} + \mu_{2,3}\mu_{5,6} \\
q_{m_{1,4},2} &= -E\mu_{2,5}\mu_{3,6} + (b-1)(c-1)\mu_{2,3}\mu_{5,6} + \eta_4\lambda_1 \\
q_{m_{1,4},3} &= -F\mu_{2,5}\mu_{3,6} + bc\mu_{2,3}\mu_{5,6} + \eta_1\lambda_4
\end{aligned}$$

$$\begin{aligned}
q_{m_{2,4},1} &= \mu_{3,5}\mu_{1,6} - \mu_{1,5}\mu_{3,6} + \mu_{1,3}\mu_{5,6} \\
q_{m_{2,4},2} &= E\mu_{1,5}\mu_{3,6} + (a-b)(c-1)\mu_{1,3}\mu_{5,6} + \eta_4\lambda_2 \\
q_{m_{2,4},3} &= (b-d)\mu_{1,5}\mu_{3,6} - b\mu_{1,3}\mu_{5,6} + \eta_2\lambda_4
\end{aligned}$$

$$\begin{aligned}
q_{m_{3,4},1} &= \mu_{2,5}\mu_{1,6} - \mu_{1,5}\mu_{2,6} + \mu_{1,2}\mu_{5,6} \\
q_{m_{3,4},2} &= -E\mu_{1,5}\mu_{2,6} + (a-b)(1-d)\mu_{1,2}\mu_{5,6} + \eta_4\lambda_3 \\
q_{m_{3,4},3} &= (c-a)\mu_{1,5}\mu_{2,6} + a\mu_{1,2}\mu_{5,6} + \eta_3\lambda_4
\end{aligned}$$

$$\begin{aligned}
q_{m_{1,5},1} &= \mu_{3,4}\mu_{2,6} - \mu_{2,4}\mu_{3,6} + \mu_{2,3}\mu_{4,6} \\
q_{m_{1,5},2} &= -E\mu_{2,4}\mu_{3,6} + (a-c)(1-b)\mu_{2,3}\mu_{4,6} + \eta_5\lambda_1 \\
q_{m_{1,5},3} &= (d-c)\mu_{2,4}\mu_{3,6} + c\mu_{2,3}\mu_{4,6} + \eta_1\lambda_5
\end{aligned}$$

$$\begin{aligned}
q_{m_{2,5},1} &= \mu_{3,4}\mu_{1,6} - \mu_{1,4}\mu_{3,6} + \mu_{1,3}\mu_{4,6} \\
q_{m_{2,5},2} &= aE\mu_{1,4}\mu_{3,6} + (a-b)(c-a)\mu_{1,3}\mu_{4,6} + \eta_5\lambda_2 \\
q_{m_{2,5},3} &= (1-d)\mu_{1,4}\mu_{3,6} - \mu_{1,3}\mu_{4,6} + \eta_2\lambda_5
\end{aligned}$$

$$\begin{aligned}
q_{m_{3,5},1} &= \mu_{2,4}\mu_{1,6} - \mu_{1,4}\mu_{2,6} + \mu_{1,2}\mu_{4,6} \\
q_{m_{3,5},2} &= -bE\mu_{1,4}\mu_{2,6} + (a-b)(b-d)\mu_{1,2}\mu_{4,6} + \eta_5\lambda_3 \\
q_{m_{3,5},3} &= (c-1)\mu_{1,4}\mu_{2,6} + \mu_{1,2}\mu_{4,6} + \eta_3\lambda_5
\end{aligned}$$

$$\begin{aligned}
q_{m_{4,5},1} &= \mu_{2,3}\mu_{1,6} - \mu_{1,3}\mu_{2,6} + \mu_{1,2}\mu_{3,6} \\
q_{m_{4,5},2} &= b(c-a)\mu_{1,3}\mu_{2,6} + a(b-d)\mu_{1,2}\mu_{3,6} + \eta_5\lambda_4 \\
q_{m_{4,5},3} &= (c-1)\mu_{1,3}\mu_{2,6} + (1-d)\mu_{1,2}\mu_{3,6} + \eta_4\lambda_5
\end{aligned}$$

$$\begin{aligned}
q_{m_{1,6},1} &= \mu_{3,4}\mu_{2,5} - \mu_{2,4}\mu_{3,5} + \mu_{2,3}\mu_{4,5} \\
q_{m_{1,6},2} &= -E\mu_{2,4}\mu_{3,5} + (a-c)(1-d)\mu_{2,3}\mu_{4,5} + \eta_6\lambda_1 \\
q_{m_{1,6},3} &= (b-a)\mu_{2,4}\mu_{3,5} + a\mu_{2,3}\mu_{4,5} + \eta_1\lambda_6
\end{aligned}$$

$$\begin{aligned}
q_{m_{2,6},1} &= \mu_{3,4}\mu_{1,5} - \mu_{1,4}\mu_{3,5} + \mu_{1,3}\mu_{4,5} \\
q_{m_{2,6},2} &= cE\mu_{1,4}\mu_{3,5} + (a-c)(d-c)\mu_{1,3}\mu_{4,5} + \eta_6\lambda_2 \\
q_{m_{2,6},3} &= (1-b)\mu_{1,4}\mu_{3,5} - \mu_{1,3}\mu_{4,5} + \eta_2\lambda_6
\end{aligned}$$

$$\begin{aligned}
q_{m_{3,6},1} &= \mu_{2,4}\mu_{1,5} - \mu_{1,4}\mu_{2,5} + \mu_{1,2}\mu_{4,5} \\
q_{m_{3,6},2} &= -dE\mu_{1,4}\mu_{2,5} + (d-b)(d-c)\mu_{1,2}\mu_{4,5} + \eta_6\lambda_3 \\
q_{m_{3,6},3} &= (a-1)\mu_{1,4}\mu_{2,5} + \mu_{1,2}\mu_{4,5} + \eta_3\lambda_6
\end{aligned}$$

$$\begin{aligned}
q_{m_{4,6},1} &= \mu_{2,3}\mu_{1,5} - \mu_{1,3}\mu_{2,5} + \mu_{1,2}\mu_{3,5} \\
q_{m_{4,6},2} &= d(c-a)\mu_{1,3}\mu_{2,5} + c(b-d)\mu_{1,2}\mu_{3,5} + \eta_6\lambda_4 \\
q_{m_{4,6},3} &= (a-1)\mu_{1,3}\mu_{2,5} + (1-b)\mu_{1,2}\mu_{3,5} + \eta_4\lambda_6
\end{aligned}$$

$$\begin{aligned}
q_{m_{5,6},1} &= \mu_{2,3}\mu_{1,4} - \mu_{1,3}\mu_{2,4} + \mu_{1,2}\mu_{3,4} \\
q_{m_{5,6},2} &= d(c-1)\mu_{1,3}\mu_{2,4} + c(1-d)\mu_{1,2}\mu_{3,4} + \eta_6\lambda_5 \\
q_{m_{5,6},3} &= b(a-1)\mu_{1,3}\mu_{2,4} + a(1-b)\mu_{1,2}\mu_{3,4} + \eta_5\lambda_6
\end{aligned}$$

$$\begin{aligned}
q_{E_1,1} &= (d-b)/E\mu_{1,2}\lambda_2 + (c-a)/E\mu_{1,3}\lambda_3 + \mu_{1,4}\lambda_4 \\
q_{E_1,2} &= (d-1)/E\mu_{1,2}\lambda_2 + (c-1)/E\mu_{1,3}\lambda_3 + \mu_{1,5}\lambda_5 \\
q_{E_1,3} &= (b-1)/E\mu_{1,2}\lambda_2 + (a-1)/E\mu_{1,3}\lambda_3 + \mu_{1,6}\lambda_6
\end{aligned}$$

$$\begin{aligned}
q_{E_2,1} &= F/E\mu_{1,2}\lambda_1 + (c-a)/E\mu_{2,3}\lambda_3 + \mu_{2,4}\lambda_4 \\
q_{E_2,2} &= (c-d)/E\mu_{1,2}\lambda_1 + (c-1)/E\mu_{2,3}\lambda_3 + \mu_{2,5}\lambda_5 \\
q_{E_2,3} &= (a-b)/E\mu_{1,2}\lambda_1 + (a-1)/E\mu_{2,3}\lambda_3 + \mu_{2,6}\lambda_6
\end{aligned}$$

$$\begin{aligned}
q_{E_3,1} &= F/E\mu_{1,3}\lambda_1 + (b-d)/E\mu_{2,3}\lambda_2 + \mu_{3,4}\lambda_4 \\
q_{E_3,2} &= (c-d)/E\mu_{1,3}\lambda_1 + (1-d)/E\mu_{2,3}\lambda_2 + \mu_{3,5}\lambda_5 \\
q_{E_3,3} &= (a-b)/E\mu_{1,3}\lambda_1 + (1-b)/E\mu_{2,3}\lambda_2 + \mu_{3,6}\lambda_6
\end{aligned}$$

$$\begin{aligned}
q_{E_4,1} &= F/(a-c)\mu_{1,4}\lambda_1 + (b-d)/(a-c)\mu_{2,4}\lambda_2 + \mu_{3,4}\lambda_3 \\
q_{E_4,2} &= c/(a-c)\mu_{1,4}\lambda_1 + 1/(a-c)\mu_{2,4}\lambda_2 + \mu_{4,5}\lambda_5 \\
q_{E_4,3} &= a/(a-c)\mu_{1,4}\lambda_1 + 1/(a-c)\mu_{2,4}\lambda_2 + \mu_{4,6}\lambda_6
\end{aligned}$$

$$\begin{aligned}
q_{E_5,1} &= (d-c)/(c-1)\mu_{1,5}\lambda_1 + (d-1)/(c-1)\mu_{2,5}\lambda_2 + \mu_{3,5}\lambda_3 \\
q_{E_5,2} &= -c/(c-1)\mu_{1,5}\lambda_1 - 1/(c-1)\mu_{2,5}\lambda_2 + \mu_{4,5}\lambda_4 \\
q_{E_5,3} &= -1/(c-1)\mu_{1,5}\lambda_1 - 1/(c-1)\mu_{2,5}\lambda_2 + \mu_{5,6}\lambda_6
\end{aligned}$$

$$\begin{aligned}
q_{E_6,1} &= (b-a)/(a-1)\mu_{1,6}\lambda_1 + (b-1)/(a-1)\mu_{2,6}\lambda_2 + \mu_{3,6}\lambda_3 \\
q_{E_6,2} &= -a/(a-1)\mu_{1,6}\lambda_1 - 1/(a-1)\mu_{2,6}\lambda_2 + \mu_{4,6}\lambda_4 \\
q_{E_6,3} &= -1/(a-1)\mu_{1,6}\lambda_1 - 1/(a-1)\mu_{2,6}\lambda_2 + \mu_{5,6}\lambda_5
\end{aligned}$$

In general, the dimension  $k$  of the ambient space  $\mathbb{A}^k$  of the universal torsor is at least as large as the number of lines on the surface plus the number of exceptional curves of its desingularization, while the dimension of the universal torsor only depends on the degree of the surface, so that the number of equations must grow with  $k$ .

Heuristically, the complexity of universal torsors should be dictated by the following considerations:

- The dimension of the universal torsor of split Del Pezzo surfaces  $S$  is  $12 - d$ , where  $d$  is the degree of  $S$ .
- For smooth Del Pezzo surfaces, the number of lines is bigger in smaller degrees (e.g., 10 lines in degree 5, and 27 lines in degree 3).

- Singular surfaces have less lines than smooth surfaces.
- The number of lines is higher in cases with “few mild” singularities (e.g., for cubics:  $\mathbf{A}_1$  with 21 lines,  $\mathbf{A}_2$  with 15 lines), while it is low for “bad” singularities (e.g., 1 for the  $\mathbf{E}_6$  cubic, 2 for the  $\mathbf{A}_5 + \mathbf{A}_1$  cubic).

Therefore, we expect universal torsors over surfaces which have low degree, are smooth or have mild singularities to be more complex than torsors over surfaces in large degree, or with complicated singularities.

### References

- [BB] R. de la Bretèche and T. D. Browning. On Manin’s conjecture for singular del Pezzo surfaces of degree four, I. *Mich. Math. J.*, to appear.
- [BB05] R. de la Bretèche and T. D. Browning. On Manin’s conjecture for singular del Pezzo surfaces of degree four, II, arXiv:math.NT/0502510, 2005.
- [BBD05] R. de la Bretèche, T. D. Browning, and U. Derenthal. On Manin’s conjecture for a certain singular cubic surface, arXiv:math.NT/0509370, 2005.
- [BF04] R. de la Bretèche and É. Fouvry. L’éclaté du plan projectif en quatre points dont deux conjugués. *J. Reine Angew. Math.*, 576:63–122, 2004.
- [Bir62] B. J. Birch. Forms in many variables. *Proc. Roy. Soc. Ser. A*, 265:245–263, 1961/1962.
- [BM90] V. V. Batyrev and Y. I. Manin. Sur le nombre des points rationnels de hauteur borné des variétés algébriques. *Math. Ann.*, 286(1-3):27–43, 1990.
- [BP04] V. V. Batyrev and O. N. Popov. The Cox ring of a del Pezzo surface. In *Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002)*, volume 226 of *Progr. Math.*, pages 85–103. Birkhäuser Boston, Boston, MA, 2004.
- [Bre98] R. de la Bretèche. Sur le nombre de points de hauteur bornée d’une certaine surface cubique singulière. *Astérisque*, (251):51–77, 1998. Nombre et répartition de points de hauteur bornée (Paris, 1996).

- [Bre02] R. de la Bretèche. Nombre de points de hauteur bornée sur les surfaces de del Pezzo de degré 5. *Duke Math. J.*, 113(3):421–464, 2002.
- [Bro04] T. D. Browning. The density of rational points on a certain singular cubic surface, arXiv:math.NT/0404245, 2004.
- [Bro05] T. D. Browning. An overview of Manin’s conjecture for del Pezzo surfaces, arXiv:math.NT/0511041, 2005.
- [BT98] V. V. Batyrev and Y. Tschinkel. Manin’s conjecture for toric varieties. *J. Algebraic Geom.*, 7(1):15–53, 1998.
- [CLT02] A. Chambert-Loir and Y. Tschinkel. On the distribution of points of bounded height on equivariant compactifications of vector groups. *Invent. Math.*, 148(2):421–452, 2002.
- [CTS87] J.-L. Colliot-Thélène and J.-J. Sansuc. La descente sur les variétés rationnelles. II. *Duke Math. J.*, 54(2):375–492, 1987.
- [Der05] U. Derenthal. Manin’s conjecture for a certain singular cubic surface, arXiv:math.NT/0504016, 2005.
- [Der06a] U. Derenthal. On a constant arising in Manin’s conjecture for Del Pezzo surfaces, 2006.
- [Der06b] U. Derenthal. On the Cox ring of singular Del Pezzo surfaces, 2006.
- [Der06c] U. Derenthal. On the Cox ring of Del Pezzo surfaces, arXiv:math.AG/0603111, 2006.
- [Der06d] U. Derenthal. Singular Del Pezzo surfaces whose universal torsors are hypersurfaces, arXiv:math.AG/0604194, 2006.
- [Dol03] I. Dolgachev. *Lectures on invariant theory*, volume 296 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2003.
- [FMT89] J. Franke, Y. I. Manin, and Y. Tschinkel. Rational points of bounded height on Fano varieties. *Invent. Math.*, 95(2):421–435, 1989.
- [Har77] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.
- [HB03] D. R. Heath-Brown. The density of rational points on Cayley’s cubic surface. In *Proceedings of the Session in Analytic Number Theory and Diophantine Equations*, volume 360 of *Bonner Math. Schriften*, page 33, Bonn, 2003. Univ. Bonn.



- [HK00] Y. Hu and S. Keel. Mori dream spaces and GIT. *Michigan Math. J.*, 48:331–348, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.
- [HT04] B. Hassett and Y. Tschinkel. Universal torsors and Cox rings. In *Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002)*, volume 226 of *Progr. Math.*, pages 149–173. Birkhäuser Boston, Boston, MA, 2004.
- [Man86] Yu. I. Manin. *Cubic forms*, volume 4 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1986. Algebra, geometry, arithmetic, Translated from the Russian by M. Hazewinkel.
- [Pey95] E. Peyre. Hauteurs et mesures de Tamagawa sur les variétés de Fano. *Duke Math. J.*, 79(1):101–218, 1995.
- [Pey98] E. Peyre. Terme principal de la fonction zêta des hauteurs et torseurs universels. *Astérisque*, (251):259–298, 1998. Nombre et répartition de points de hauteur bornée (Paris, 1996).
- [Sal98] P. Salberger. Tamagawa measures on universal torsors and points of bounded height on Fano varieties. *Astérisque*, (251):91–258, 1998. Nombre et répartition de points de hauteur bornée (Paris, 1996).